Lecture 12: Whitehead Theorem and CW approximation

Definition

We define the category **TopP** of topological pairs where an object

$$(X, A)$$

is a topological space X with a subspace A, and morphisms $(X, A) \to (Y, B)$ are continuous maps

$$f : X \to Y \quad \text{such that} \quad f(A) \subset B.$$

A homotopy between two maps $f_1, f_2 : (X, A) \to (Y, B)$ is a homotopy $F : X \times I \to Y$ between f_0, f_1 such that

$$F|_{X \times t}(A) \subset B \quad \text{for any} \quad t \in I.$$

The quotient category of $\underline{\mathbf{TopP}}$ by homotopy of maps is denoted by $\underline{\mathbf{hTopP}}$. The pointed versions are defined similarly and denoted by $\underline{\mathbf{TopP}}_*$ and $\underline{\mathbf{hTopP}}_*$. Morphisms in $\underline{\mathbf{hTopP}}$ and $\underline{\mathbf{hTopP}}_*$ are denoted by

$$[(X, A), (Y, B)], \qquad [(X, A), (Y, B)]_0.$$

When we work with the convenient category $\underline{\mathscr{T}}$, we have similar notions of $\underline{\mathscr{T}\mathbf{P}}$ for a pair of spaces, $\underline{\mathrm{h}\mathscr{T}\mathbf{P}}$ for the quotient homotopy category, and $\underline{\mathscr{T}\mathbf{P}}_\star, \underline{\mathrm{h}\mathscr{T}\mathbf{P}}_\star$ for the pointed cases.

Theorem

Let $f\colon (X,A) \to (Y,B)$ in $\mathrm{h}\mathscr{T}\mathbf{P}_\star$. Let $\bar{f} = f|_A$. Then the sequence

$$(X,A) \to (Y,B) \to (C_f, C_{\bar{f}}) \to \Sigma(X,A) \to \Sigma(Y,B) \to \Sigma(C_f, C_{\bar{f}}) \to \Sigma^2(X,A) \to \cdots$$

is co-exact in $\mathrm{h}\mathscr{T}\mathbf{P}_\star$.

This generalizes the co-exact Puppe sequence to the pair case.

Definition

Let $(X, A) \in \underline{\mathscr{T}\mathbf{P}_\star}$. We define the relative homotopy group by

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]_0.$$

We will also write $\pi_n(X, A; x_0)$ when we specify the base point.

Note that

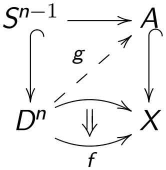
$$(D^n, S^{n-1}) \simeq \Sigma^{n-1}(D^1, S^0), \quad n \geq 2.$$

$\pi_n(X, A)$ is a group for $n \geq 2$ due to the adjunct pair (Σ, Ω).

Lemma

$f\colon (D^n, S^{n-1}) \to (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel S^{n-1} to a map whose image lies in A.

This lemma can be illustrated by the following diagram



Here g maps D^n to A and $g \simeq f$ rel S^{n-1}.

Proof

Assume $[f]_0 = 0$ in $\pi_n(X, A)$. Then we can find a homotopy

$$F : D^n \times I \to X \quad \text{s.t.} \quad F(-,0) = x_0, \quad F(-,t) \in A, \quad F(-,1) = f(-).$$

Let us view the restriction of F to $S^{n-1} \times I \cup D^n \times \{0\}$ as defining a map (via a natural homeomorphism)

$$g : (D^n, S^{n-1}) \to (X, A).$$

Then F can be viewed as a homotopy $g \simeq f$ rel S^{n-1} as required. Conversely, assume there exists $g : (D^n, S^{n-1}) \to (X, A)$ such that $g \simeq f$ rel S^{n-1}. Let

$$F : D^n \times I \to D^n$$

be a homotopy from the identity to the trivial map. Then

$$F \circ g : D^n \times I \to X$$

shows that $[g]_0 = 0$, hence $[f]_0 = 0$ as well.

Theorem

Let $A \subset X$ in $\mathscr{T}_\star$. Then there is a long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

Here the boundary map ∂ sends $\varphi \in [(D^n, S^{n-1}), (X, A)]_0$ to its restriction to S^{n-1}.

Proof

Consider

$$f\colon (S^0,\{0\}) \to (S^0, S^0).$$

Let $\bar{f} = f|_{\{0\}} : \{0\} \to S^0$. It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, S^0).$$

Since $\Sigma^n(S^0) = S^n, \Sigma(D^n, S^{n-1}) = (D^{n+1}, S^n)$, the co-exact sequence

$$(S^0,\{0\})\to(S^0,S^0)\to(D^1,S^0)\to(S^1,\{0\})\to(S^1,S^1)\to(D^2,S^1)\to(S^2,\{0\})\to\cdots$$

implies the exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

□

Definition

A pair (X, A) is called n-connected $(n \geq 0)$ if $\pi_0(A) \to \pi_0(X)$ is surjective and

$$\pi_k(X, A; x_0) = 0 \quad \forall 1 \leq k \leq n, x_0 \in A.$$

From the long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \cdots \to \pi_0(X)$$

we see that (X,A) is n-connected if and only if for any $x_0 \in A$

$$\begin{cases} \pi_r(A,x_0) \to \pi_r(X,x_0) \text{ is bijective for } r < n \\ \pi_n(A,x_0) \to \pi_n(X,x_0) \text{ is surjective} \end{cases}$$

Definition

A map $f\colon X \to Y$ is called an n-equivalence $(n \geq 0)$ if for any $x_0 \in X$

$$\begin{cases} f_* : \pi_r(X, x_0) \to \pi_r(Y, f(x_0)) \text{ is bijective for } r < n \\ f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \text{ is surjective} \end{cases}$$

f is called weak homotopy equivalence or ∞-equivalence if f is n-equivalence for any $n \geq 0$.

Example

For any $n \geq 0$, the pair (D^{n+1}, S^n) is n-connected.

Whitehead Theorem

Lemma

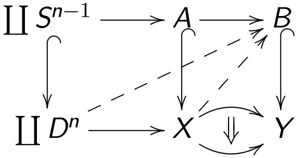
Let X be obtained from A by attaching n-cells. Let (Y, B) be a pair such that

$$\begin{cases} \pi_n(Y, B; b) = 0, \forall b \in B & \text{if} \quad n \geq 1 \\ \pi_0(B) \to \pi_0(Y) \text{ is surjective} & \text{if} \quad n = 0. \end{cases}$$

Then any map from $(X, A) \to (Y, B)$ is homotopic rel A to a map from X to B.

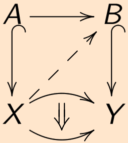
Proof.

This follows from the universal property of push-out



Theorem

Let (X, A) be a relative CW complex with relative dimension $\leq n$. Let (Y, B) be n-connected $(0 \leq n \leq \infty)$. Then any map from (X, A) to (Y, B) is homotopic relative to A to a map from X to B.



Proof.

Apply the previous Lemma to

$$A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and observe that all embeddings are cofibrations. □

Proposition

Let $f\colon X \to Y$ be a weak homotopy equivalence, P be a CW complex. Then

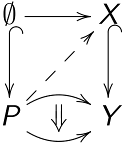
$$f_* : [P, X] \to [P, Y]$$

is a bijection.

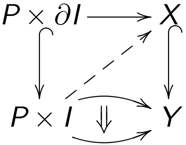
Proof

We can assume f is an embedding and (Y, X) is ∞-connected. Otherwise replace Y by M_f.

Surjectivity is illustrated the diagram (applying previous Theorem to the pair $(P, \emptyset)$)



Injectivity is illustrated by the diagram (observing $P \times I, P \times \partial I$ are CW complexes)



Theorem (Whitehead Theorem)

A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.

Proof

Let $f: X \to Y$ be a weak homotopy equivalence between CW complexes. Apply previous Prop to $P = X, Y$, we find bijections

$$f_* : [X, X] \to [X, Y], \quad f_* : [Y, X] \to [Y, Y].$$

Let $g \in [Y, X]$ such that $f_*[g] = 1_Y$. Then $f \circ g \simeq 1_Y$.

On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] \simeq [f \circ 1] = [f] = f_*[1_X].$$

We conclude $[g \circ f] = 1_X$. Therefore f is a homotopy equivalence. The reverse direction is obvious. □

Cellular approximation

Definition

Let (X, Y) be CW complexes. A map $f : X \to Y$ is called cellular if $f(X^n) \subset Y^n$ for any n. We define the category $\underline{\mathrm{CW}}$ whose objects are CW complexes and morphisms are cellular maps.

Definition

A cellular homotopy between two cellular maps $X \to Y$ of CW complexes is a homotopy $X \times I \to Y$ that is itself a cellular map. Here I is naturally a CW complex. We define the quotient category $\underline{\mathrm{hCW}}$ of $\underline{\mathrm{CW}}$ whose morphisms are cellular homotopy class of cellular maps.

Lemma

Let X be obtained from A by attaching n-cells $(n \geq 1)$, then (X, A) is $(n-1)$-connected.

Corollary

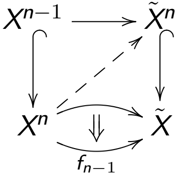
Let (X, A) be a relative CW complex, then for any $n \geq 0$, the pair (X, X^n) is n-connected.

Theorem

Let $f : (X, A) \to (\tilde{X}, \tilde{A})$ between relative CW complexes which is cellular on a subcomplex (Y, B) of (X, A). Then f is homotopic rel Y to a cellular map $g : (X, A) \to (\tilde{X}, \tilde{A})$.

Proof

Assume we have constructed $f_{n-1} : (X, A) \to (\tilde{X}, \tilde{A})$ which is homotopic to f rel Y and cellular on the $(n-1)$-skeleton X^{n-1}. Since $(\tilde{X}, \tilde{X}^n)$ is n-connected,



we can find a homotopy rel X^{n-1} from $f_{n-1}|_{X^n} : X^n \to \tilde{X}$ to a map $X^n \to \tilde{X}^n$. Since f is cellular on Y, we can choose this homotopy rel Y by adjusting only those n-cells not in Y. This homotopy extends to a homotopy rel $X^{n-1} \cup Y$ from f_{n-1} to a map $f_n : X \to \tilde{X}$ since $X^n \subset X$ is a cofibration. Then f_∞ works.

Theorem (Cellular Approximation Theorem)

Any map between relative CW complexes is homotopic to a cellular map. If two cellular maps between relative CW complexes are homotopic, then they are cellular homotopic.

Proof.
Apply the previous Theorem to $(X, \emptyset)$ and $(X \times I, X \times \partial I)$. $\square$

CW approximation

Definition

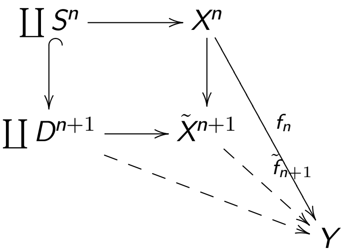
A CW approximation of a topological space Y is a CW complex X with a weak homotopy equivalence $f\colon X \to Y$.

Theorem

Any space has a CW approximation.

Proof

We may assume Y is path connected. We construct a CW approximation X of Y by induction on the skeleton X^n. Assume we have constructed $f_n : X^n \to Y$ which is an n-equivalence. We attach an $(n+1)$-cell to every generator of $\ker(\pi_n(X^n) \to \pi_n(Y))$ to obtain $\tilde{X}^{n+1}$. We can extend f_n to a map $\tilde{f}_{n+1} : \tilde{X}^{n+1} \to Y$



Since $(\tilde{X}^{n+1}, X^n)$ is also n-connected, $\tilde{f}_{n+1}$ is an n-equivalence. By construction and the surjectivity of $\pi_n(X^n) \to \pi_n(\tilde{X}^{n+1})$, $\tilde{f}_{n+1}$ defines also an isomorphism for $\pi_n(\tilde{X}^{n+1}) \to \pi_n(Y)$.

Now for every generator S_α^{n+1} of $\operatorname{coker}(\pi_{n+1}(\tilde{X}^{n+1}) \to \pi_{n+1}(Y))$, we take a wedge sum to obtain

$$X^{n+1} = \tilde{X}^{n+1} \vee (\vee_\alpha S^{n+1}).$$

Then the induced map $f_{n+1} : X^{n+1} \to Y$ extends f_n to an $(n+1)$-equivalence. Inductively we obtain a weak homotopy equivalence $f_\infty : X = X^\infty \to Y$. $\square$

Theorem

Let $f\colon X \to Y$. Let $\Gamma X \to X$, and $\Gamma Y \to Y$ be CW approximations. Then there exists a unique map in $[\Gamma X, \Gamma Y]$ making the following diagram commutes in **hTop**

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Proof.

Weak homotopy equivalence of $\Gamma Y \to Y$ implies the bijection $[\Gamma_X, \Gamma_Y] \to [\Gamma_X, Y]$.

□

Definition

Two spaces X_1, X_2 are said to have the same weak homotopy type if there exists a space Y and weak homotopy equivalences $f_i : Y \to X_i, i = 1, 2$.

Theorem

Weak homotopy type is an equivalence relation.